

# On homology of map spaces

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## Abstract

Following an idea of Bendersky–Gitler, we construct an isomorphism between Anderson’s and Arone’s complexes modelling the chain complex of a map space. This allows us to apply Shipley’s convergence theorem to Arone’s model. As a corollary, we reduce the problem of homotopy equivalence for certain “toy” spaces to a problem in homological algebra.

A *space* is a pointed simplicial set. A *map* is a basepoint-preserving simplicial map. Chains, homology etc. are reduced with coefficients in a commutative ring  $R$ .

Fix spaces  $X$  and  $Y$ . We are interested in the homology of  $Y^X$ , the space of maps  $X \rightarrow Y$ .

**0.A. Arone’s approach.** Let  $\Omega$  be the category whose objects are the sets  $\langle s \rangle = \{1, \dots, s\}$ ,  $s > 0$ , and whose morphisms are surjective functions. Let  $\Omega^\circ$  denote the dual category. For  $n \in \mathbf{Z}$ , let us define a functor  $M_n(X): \Omega^\circ \rightarrow R\text{-Mod}$ . Set  $M_n(X)(\underline{s}) = C_n(X^{\wedge s})$ , where  $X^{\wedge s}$  is the  $s$ th smash power. For a morphism  $h: \langle t \rangle \rightarrow \langle s \rangle$ , set  $M_n(X)(h) = C_n(h^\sharp): C_n(X^{\wedge s}) \rightarrow C_n(X^{\wedge t})$ , where the map  $h^\sharp: X^{\wedge s} \rightarrow X^{\wedge t}$  is given by  $h^\sharp(x_1 \dots x_s) = x_{h(1)} \dots x_{h(t)}$  for  $x_1, \dots, x_s \in X_n$ ,  $n \geq 0$ . Here the simplex  $x_1 \dots x_s \in (X^{\wedge s})_n$  is the image of the simplex  $(x_1, \dots, x_s) \in (X^s)_n$  under the projection.

**0.1. Lemma.** *The functors  $M_n(X)$  are projective objects of the abelian category of functors  $\Omega^\circ \rightarrow R\text{-Mod}$ .*

Proof is given in 1.B.

The boundary operators  $\partial: C_n(X^{\wedge s}) \rightarrow C_{n-1}(X^{\wedge s})$  form a functor morphism  $\partial: M_n(X) \rightarrow M_{n-1}(X)$ . Thus  $M_*(X)$  is a chain complex of functors.

**0.2. Corollary.** *If a map  $e: X \rightarrow Y$  is a weak equivalence, then the induced chain homomorphism  $M_*(e): M_*(X) \rightarrow M_*(Y)$  is a chain homotopy equivalence.*  $\square$

We have the (unbounded) chain complex of  $R$ -modules

$$G_*(X, Y) = \text{Hom}_*(M_*(X), M_*(Y))$$

and a chain homomorphism

$$\lambda_*(X, Y): C_*(Y^X) \rightarrow G_*(X, Y),$$

see 2.C, 2.D. A natural filtration of  $G_*(X, Y)$  yields the Arone spectral sequence

$$H_{t-s}(\mathrm{Hom}_{\Sigma_s} (C_*(X^{(s)}), C_*(Y^{\wedge s}))) = {}^1E_t^s \Rightarrow H_{t-s}(G_*(X, Y)), \quad (*)$$

where  $X^{(s)} = X^{\wedge s}/(\text{fat diagonal})$  [4], [1]. [6, Theorem 9.2] ensures conditional convergence. If  $Y$  is  $(\dim X)$ -connected, then the convergence is strong and  $\lambda_*(X, Y)$  is a quasi-isomorphism, see [4] for the precise statement. (A similar result was obtained in [11, Ch. III, § 5].) We wish to get free of the connectivity assumption.

**0.B. Main results.** Here we suppose  $R = \mathbf{Z}/\ell$ ,  $\ell$  a prime. We call  $Y$   $\ell$ -toy if  $\pi_0(Y)$  is finite and  $\pi_n(Y, y)$  is a finite  $\ell$ -group for all  $y \in Y_0$  and  $n > 0$ .

**0.3. Theorem.** *Suppose that  $X$  is essentially compact<sup>1</sup> and  $Y$  is fibrant and  $\ell$ -toy. Then  $\lambda_*(X, Y)$  is a quasi-isomorphism.*

This follows from Theorems 0.5 and 0.6 below, see § 4 for details. Under the assumptions of the theorem, the convergence of  $(*)$  is strong by [6, Theorem 7.1].

**0.4. Corollary.** *Suppose that  $X$  and  $Y$  are essentially compact and  $\ell$ -toy. Suppose that the complexes  $M_*(X)$  and  $M_*(Y)$  are chain homotopy equivalent. Then  $X$  and  $Y$  are weakly equivalent.*

The proof is given in § 5. There seems to be no easy/functorial way to extract  $\pi_1(X)$  or the ring structure of  $H^*(X)$  from  $M_*(X)$ . The corollary has an algebraic analogue [9].

**0.C. Anderson's approach.** For a pointed set  $S$ , the space  $Y^S$  is defined to be the fibre of the projection

$$\prod_{s \in S} Y \rightarrow Y$$

corresponding to  $s = *$  (this agrees with our convention that maps preserve basepoints).

We have an (unbounded) chain complex  $D_*(X, Y)$  with

$$D_n(X, Y) = \prod_{q-p=n} C_q(Y^{X^p})$$

and a chain homomorphism

$$\mu_*(X, Y): C_*(Y^X) \rightarrow D_*(X, Y),$$

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<sup>1</sup>A space is *compact* (or *finite*) if it is generated by a finite number of simplices. *Essentially compact* means weakly equivalent to a compact space.

see 2.F, 2.G for details. A natural filtration of  $D_*(X, Y)$  yields the Anderson spectral sequence

$$H_q(Y^{X_p}) = {}^1E_q^p \Rightarrow H_{q-p}(D_*(X, Y)).$$

If  $Y$  is  $(\dim X)$ -connected, then  $\mu_*(X, Y)$  is a quasi-isomorphism, see [2] and [7, 4.2] for precise statements. Shipley got rid of the connectivity assumption [10].

**0.5. Theorem.** *Suppose that  $R = \mathbf{Z}/\ell$ ,  $\ell$  a prime. Suppose that  $X$  is compact and  $Y$  is fibrant and  $\ell$ -toy. Then  $\mu_*(X, Y)$  is a quasi-isomorphism.*

This is a special case of Shipley's strong convergence theorem, see § 3 for details.

**0.D. Comparing  $G_*(X, Y)$  and  $D_*(X, Y)$ .** We construct a chain homomorphism

$$\epsilon_*(X, Y): D_*(X, Y) \rightarrow G_*(X, Y)$$

such that the diagram

$$\begin{array}{ccc} & & D_*(X, Y) \\ & \nearrow \mu_*(X, Y) & \downarrow \epsilon_*(X, Y) \\ C_*(Y^X) & & G_*(X, Y) \\ & \searrow \lambda_*(X, Y) & \end{array}$$

is commutative, see 2.H.

**0.6. Theorem.** *Suppose that  $X$  is gradual<sup>2</sup>. Then  $\epsilon_*(X, Y)$  is an isomorphism.*

Proof is given in 2.I.

*Remark.* In some cases, the  ${}^2E$  term of the Anderson spectral sequence [5, Theorem 7.1 (2)] and the  ${}^1E$  term of the Arone spectral sequence differ in the grading only. This suggested relation of the two approaches [1, footnote 1] and motivated this work. Our construction of  $\epsilon_*(X, Y)$  follows the line of [5, § 6].

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<sup>2</sup>A space  $X$  is *gradual* (or *finite type*) if the sets  $X_n$ ,  $n \geq 0$ , are finite.

## 1. Preliminaries

**1.A. Notation.** For a pointed set  $S$ , we put  $S^\times = S \setminus \{*\}$ .

$\Delta_+^p$  is the standard  $p$ -simplex with an added basepoint. Let  $\iota_p \in (\Delta_+^p)_p$  be the fundamental simplex.

For  $x \in X_n$ ,  $[x] \in C_n(X)$  is the chain consisting of the single simplex  $x$  with the coefficient 1.

Given functors  $F, F': \Omega^\circ \rightarrow R\text{-Mod}$ , a functor morphism  $T: F \rightarrow F'$  consists of homomorphisms  ${}^sT: F(\langle s \rangle) \rightarrow F'(\langle s \rangle)$ .

**1.B. Proof of Lemma 0.1** (cf. [3, § I]). Fix a linear order on  $X_n^\times$ . Introduce the set

$$I = \coprod_{s \geq 0} \{(x_1, \dots, x_s) \mid x_1, \dots, x_s \in X_n^\times, x_1 < \dots < x_s\}.$$

For  $i = (x_1, \dots, x_s) \in I$ , put  $|i| = s$  and  $e_i = [x_1 \dots x_s] \in C_n(X^{\wedge s}) = M_n(X)(\langle s \rangle)$ . The elements  $e_i$  form a basis of  $M_n(X)$  in the following sense. For any functor  $F: \Omega^\circ \rightarrow R\text{-Mod}$  and elements  $a_i \in F(|i|)$ ,  $i \in I$ , there exists a unique functor morphism  $T: M_n(X) \rightarrow F$  such that  $|i|T(e_i) = a_i$  for all  $i \in I$ . Therefore, for a functor epimorphism  $\tilde{F} \rightarrow F$ , any functor morphism  $M_n(X) \rightarrow F$  lifts to  $\tilde{F}$ .  $\square$

## 2. Main constructions

**2.A. Diagonal complexes.** A bicomplex  $\underline{W}_*^*$  (of  $R$ -modules) has differentials  $d': \underline{W}_q^{p-1} \rightarrow \underline{W}_q^p$  and  $d'': \underline{W}_q^p \rightarrow \underline{W}_{q-1}^p$ , which commute:  $d''d' = d'd''$ . The diagonal (or complete total) chain complex  $\text{diag}_* \underline{W}_*^* = W_*$  of  $\underline{W}_*^*$  has

$$W_n = \prod_{q-p=n} \underline{W}_q^p.$$

For  $w \in W_n$ , we have  $w = (w_q^p)_{q-p=n}$ , where  $w_q^p \in \underline{W}_q^p$ . The differential  $\partial: W_n \rightarrow W_{n-1}$  is defined by

$$(\partial w)_q^p = d''(w_{q+1}^p) - (-1)^n d'(w_q^{p-1}), \quad q-p = n-1.$$

**2.B. The complex  $\text{Hom}_*(U_*, V_*)$ .** Given chain complexes  $U_*$  and  $V_*$  in some  $R$ -linear category, we define the bicomplex  $\underline{\text{Hom}}_*^*(U_*, V_*)$  with  $\underline{\text{Hom}}_q^p(U_*, V_*) = \text{Hom}(U_p, V_q)$  and the differentials induced by those of  $U_*$  and  $V_*$ . We have

$$\text{Hom}_*(U_*, V_*) = \text{diag}_* \underline{\text{Hom}}_*^*(U_*, V_*).$$

**2.C. The complex  $G_*(X, Y)$ .** We put

$$\underline{G}_*(X, Y) = \underline{\text{Hom}}_*^*(M_*(X), M_*(Y)), \quad G_*(X, Y) = \text{Hom}_*(M_*(X), M_*(Y)).$$

**2.D. Construction of  $\lambda_*(X, Y)$ .** For  $s > 0$ , let  ${}^s\eta: Y^X \wedge X^{\wedge s} \rightarrow Y^{\wedge s}$  be the evaluation map. For  $s > 0$  and  $p, q \in \mathbf{Z}$ , we have the homomorphism  $C_q({}^s\eta): C_q(Y^X \wedge X^{\wedge s}) \rightarrow C_q(Y^{\wedge s})$  and define the homomorphism

$${}^s\lambda_q^p: C_{q-p}(Y^X) \rightarrow \text{Hom}(C_p(X^{\wedge s}), C_q(Y^{\wedge s}))$$

by

$${}^s\lambda_q^p(z)(u) = C_q({}^s\eta)(z \times u), \quad u \in C_p(X^{\wedge s}), \quad z \in C_{q-p}(Y^X).$$

The homomorphisms  ${}^s\lambda_q^p$  form the promised chain homomorphism  $\lambda_*(X, Y)$ .

**2.E. The complex  $D_*(V)$ .** For a cosimplicial space  $V$ , we have the bicomplex  $\underline{D}_*(V)$  with  $\underline{D}_q^p(V) = C_q(V^p)$  and the following differentials. The differential  $d': C_q(V^{p-1}) \rightarrow C_q(V^p)$  is defined by

$$d' = \sum_{i=0}^p (-1)^i C_q(\delta^i),$$

where  $\delta^i: V^{p-1} \rightarrow V^p$  are the coface maps. The differential  $d'': C_q(Y^{X_p}) \rightarrow C_{q-1}(Y^{X_p})$  is the ordinary boundary operator. We put  $D_*(V) = \text{diag}_* \underline{D}_*(V)$ .

**2.F. The complex  $D_*(X, Y)$ .** Consider the cosimplicial space  $V = \text{hom}(X, Y)$  with  $V^p = Y^{X_p}$  [8, Ch. X, 2.2 (ii)]. We put

$$\underline{D}_*(X, Y) = \underline{D}_*(V), \quad D_*(X, Y) = D_*(V).$$

**2.G. Construction of  $\mu_*(X, Y)$ .** For  $x \in X_p$ , we have the composite map

$$\theta^x: Y^X \wedge \Delta_+^p \xrightarrow{\text{id} \wedge \bar{x}} Y^X \wedge X \xrightarrow{\eta} Y,$$

where  $\bar{x}: \Delta_+^p \rightarrow X$  is the characteristic map of the simplex  $x$  and  $\eta$  is the evaluation map. Combining  $\theta^x$  over all  $x \in X_p$ , we get a map

$$\theta^p: Y^X \wedge \Delta_+^p \rightarrow Y^{X_p}.$$

For  $p \geq 0$  and  $q \in \mathbf{Z}$ , we have the homomorphism  $C_q(\theta^p): C_q(Y^X \wedge \Delta_+^p) \rightarrow C_q(Y^{X_p})$  and introduce the homomorphism

$$\mu_q^p: C_{q-p}(Y^X) \rightarrow C_q(Y^{X_p}), \quad \mu_q^p(z) = C_q(\theta^p)(z \times [\iota_p]).$$

The homomorphisms  $\mu_q^p$  form the promised chain homomorphism  $\mu_*(X, Y)$ .

**2.H. Construction of  $\epsilon_*(X, Y)$ .** A simplex  $v \in (Y^{X_p})_q$  is a basepoint-preserving function  $v: X_p \rightarrow Y_q$ . For  $s > 0$  and  $p, q \geq 0$ , we define the homomorphism

$${}^s\epsilon_q^p: C_q(Y^{X_p}) \rightarrow \text{Hom}(C_p(X^{\wedge s}), C_q(Y^{\wedge s}))$$

by

$${}^s\epsilon_q^p([v])([x_1 \dots x_s]) = [v(x_1) \dots v(x_s)], \quad x_1, \dots, x_s \in X_p, \quad v \in (Y^{X_p})_q.$$

The homomorphisms  ${}^s\epsilon_q^p$  form a homomorphism of bicomplexes

$$\underline{\epsilon}_*^*(X, Y): \underline{D}_*^*(X, Y) \rightarrow \underline{G}_*^*(X, Y)$$

and thus the promised chain homomorphism  $\epsilon_*(X, Y)$ .

*Remark.* The bicomplexes  $\underline{D}_*^*(X, Y)$  and  $\underline{G}_*^*(X, Y)$  are in fact cosimplicial simplicial  $R$ -modules. (To see this, recall that, for every space  $Z$ ,  $C_*(Z)$  is in fact a simplicial  $R$ -module and thus  $M_*(Z)$  is a simplicial functor.) The homomorphism  $\underline{\epsilon}_*^*(X, Y)$  preserves this structure.

One easily verifies that  $\epsilon_*(X, Y) \circ \mu_*(X, Y) = \lambda_*(X, Y)$ .

**2.1. Proof of Theorem 0.6.** Take  $p, q \geq 0$ . It suffices to prove that the homomorphism

$$\epsilon_q^p = ({}^s\epsilon_q^p)_{s \geq 0}: C_q(Y^{X_p}) \rightarrow \text{Hom}(M_p(X), M_q(Y))$$

is an isomorphism. We construct a homomorphism

$$\xi_q^p: \text{Hom}(M_p(X), M_q(Y)) \rightarrow C_q(Y^{X_p})$$

and leave to the reader to verify that  $\xi_q^p \circ \epsilon_q^p$  and  $\epsilon_q^p \circ \xi_q^p$  are the identities.

Fix a linear order on  $X_p^\times$ . Suppose we are given sets  $E, F \subseteq X_p^\times$  such that  $E \supseteq F \neq \emptyset$ . We have  $E = \{x_1, \dots, x_s\}$  for some  $x_1 < \dots < x_s$ . Put  $\kappa_E = x_1 \dots x_s \in (X^{\wedge s})_p$ . For  $y_1, \dots, y_s \in Y_q$ , define the function  $\phi_E^F(y_1, \dots, y_s): X_p \rightarrow Y_q$  by the rules

$$x_t \mapsto y_t \text{ for } t = 1, \dots, s \text{ such that } x_t \in F;$$

$$x \mapsto * \text{ for all other } x \in X_p.$$

We have the homomorphism  $\Phi_E^F: C_q(Y^{\wedge s}) \rightarrow C_q(Y^{X_p})$  with  $\Phi_E^F([y_1 \dots y_s]) = [\phi_E^F(y_1, \dots, y_s)]$  for  $y_1, \dots, y_s \in Y_q^\times$ . Define the homomorphism

$$\psi_E^F: \text{Hom}_{\Sigma_s}(C_p(X^{\wedge s}), C_q(Y^{\wedge s})) \rightarrow C_q(Y^{X_p})$$

by  $\psi_E^F(t) = \Phi_E^F(t([\kappa_E]))$ . (One may note that  $\psi_E^F$  does not depend on the order on  $X_p^\times$ .) For a functor morphism  $T: M_p(X) \rightarrow M_q(Y)$ , we set

$$\xi_q^p(T) = \sum_{E, F \subseteq X_p^\times: E \supseteq F \neq \emptyset} (-1)^{|E|-|F|} \psi_E^F(|E|T).$$

□

### 3. Anderson's model

**3.A. General cosimplicial case.** We follow [7, § 2]. Let  $V$  be a cosimplicial space. We have the (unbounded) chain complex  $D_*(V)$  (see 2.E). There is the chain homomorphism

$$\mu_*(V): C_*(\text{Tot } U) \rightarrow D_*(V)$$

formed by the homomorphisms

$$\mu_q^p: C_{q-p}(\text{Tot } U) \rightarrow C_q(V^p)$$

that are defined in the following way. A simplex  $w \in (\text{Tot } V)_n$  is a sequence  $(w^p)_{p \geq 0}$  of maps  $w^p: \Delta_+^n \wedge \Delta_+^p \rightarrow V^p$ . For  $w \in (\text{Tot } U)_{q-p}$ , we have the homomorphism  $C_q(w^p): C_q(\Delta_+^{q-p} \wedge \Delta_+^p) \rightarrow C_q(V^p)$  and set

$$\mu_q^p([w]) = C_q(w^p)([\iota_{q-p}] \times [\iota_p]).$$

**3.1. Theorem.** *Suppose that  $R = \mathbf{Z}/\ell$ ,  $\ell$  a prime,  $V$  is fibrant and the spaces  $V^p$ ,  $p \geq 0$ , and  $\text{Tot } V$  are  $\ell$ -toy. Then  $\mu_*(V)$  is a quasi-isomorphism.*

*Proof.* Apply Shipley's strong convergence theorem [10, Theorem 6.1] and [7, Lemma 2.3].  $\square$

**3.B. Proof of Theorem 0.5.** We have the cosimplicial space  $V = \text{hom}(X, Y)$  and the canonical isomorphism  $Y^X = \text{Tot } V$  [8, Ch. X, 3.3 (i)]. The diagram

$$\begin{array}{ccc} C_*(Y^X) & \xrightarrow{\mu_*(X, Y)} & D_*(X, Y) \\ \parallel & & \parallel \\ C_*(\text{Tot } V) & \xrightarrow{\mu_*(V)} & D_*(V) \end{array}$$

is commutative.

The cosimplicial space  $V$  is fibrant by [8, Ch. X, 4.7 (ii)]. The spaces  $V^p$  are  $\ell$ -toy since  $X$  is gradual and  $Y$  is  $\ell$ -toy. The spaces  $Y^X$  and thus  $\text{Tot } V$  are  $\ell$ -toy since  $X$  is compact and  $Y$  is fibrant and  $\ell$ -toy. By Theorem 3.1,  $\mu_*(V)$  is a quasi-isomorphism.  $\square$

### 4. Arone's model

#### 4.A. Homotopy invariance.

**4.1. Lemma.** *Let  $e: X' \rightarrow X$  and  $f: Y \rightarrow Y'$  be weak equivalences of spaces. Suppose that  $Y$  and  $Y'$  are fibrant. Then  $\lambda_*(X, Y)$  is a quasi-isomorphism if and only if  $\lambda_*(X', Y')$  is.*

*Proof.* The maps  $e$  and  $f$  induce a map  $g: Y^X \rightarrow Y'^{X'}$ . We have the commutative diagram

$$\begin{array}{ccc} C_*(Y^X) & \xrightarrow{\lambda_*(X,Y)} & G_*(X,Y) \\ C_*(g) \downarrow & & \downarrow G_*(e,f) \\ C_*(Y'^{X'}) & \xrightarrow{\lambda_*(X',Y')} & G_*(X',Y'). \end{array}$$

$C_*(g)$  is a quasi-isomorphism since  $g$  is a weak equivalence. It follows from Lemma 0.2 that  $G_*(e,f)$  is a quasi-isomorphism. The desired equivalence is clear now.  $\square$

**4.B. Proof of Theorem 0.3.** If  $X$  is compact, the assertion follows immediately from Theorems 0.5 and 0.6. In general,  $X$  is weakly equivalent to a compact space  $X^\circ$ . Using Lemma 4.1, we pass from  $\lambda_*(X^\circ, Y)$  to  $\lambda_*(X, Y)$ .  $\square$

## 5. Reconstructing $X$ from $M_*(X)$

### 5.A. Composition of maps and homomorphisms.

**5.1. Lemma.** Let  $X, Y$  and  $Z$  be spaces and  $\gamma: Z^Y \wedge Y^X \rightarrow Z^X$  be the composition map. Then the diagram of chain complexes and chain homomorphisms

$$\begin{array}{ccccc} C_*(Z^Y) \otimes C_*(Y^X) & \xrightarrow{\text{cross product}} & C_*(Z^Y \wedge Y^X) & \xrightarrow{C_*(\gamma)} & C_*(Z^X) \\ \downarrow \lambda_*(Y,Z) \otimes \lambda_*(X,Y) & & & & \downarrow \lambda_*(X,Z) \\ G_*(Y,Z) \otimes G_*(X,Y) & \xrightarrow{\text{composition}} & & & G_*(X,Z) \end{array}$$

is commutative.

This follows from the associativity of the cross product.  $\square$

**5.B. Proof of Corollary 0.4.** Lemma 0.2 allows us to assume  $X$  and  $Y$  fibrant. Note that  $H_0(G_*(X, Y)) = [M_*(X), M_*(Y)]$ , the  $R$ -module of chain homotopy classes. By Lemma 5.1, we have the commutative diagram

$$\begin{array}{ccccc} H_0(X^Y) \otimes H_0(Y^X) & \xrightarrow{\text{cross product}} & H_0(X^Y \wedge Y^X) & \xrightarrow{H_0(\gamma)} & H_0(X^X) \\ \downarrow H_0(\lambda_*(Y,X)) \otimes H_0(\lambda_*(X,Y)) & & & & \downarrow H_0(\lambda_*(X,X)) \\ [M_*(Y), M_*(X)] \otimes [M_*(X), M_*(Y)] & \xrightarrow{\text{composition}} & & & [M_*(X), M_*(X)], \end{array}$$

where  $\gamma: X^Y \wedge Y^X \rightarrow X^X$  is the composition map. We use the notation  $B \otimes A \mapsto B \circ A$  for the upper line homomorphism  $H_0(X^Y) \otimes H_0(Y^X) \rightarrow H_0(X^X)$ . By Theorem 0.3,  $H_0(\lambda_*(X, Y))$ ,  $H_0(\lambda_*(Y, X))$  and  $H_0(\lambda_*(X, X))$  are isomorphisms.



Let  $f: M_*(X) \rightarrow M_*(Y)$  and  $g: M_*(Y) \rightarrow M_*(X)$  be mutually inverse chain homotopy equivalences. We have  $[f] = H_0(\lambda_*(X, Y))(A)$  for some  $A \in H_0(Y^X)$  and  $[g] = H_0(\lambda_*(Y, X))(B)$  for some  $B \in H_0(X^Y)$ . By the diagram,  $B \circ A = 1$  in  $H_0(X^X)$ . Thus there are maps  $a: X \rightarrow Y$  and  $b: Y \rightarrow X$  such that  $b \circ a \sim \text{id}_X$ . Interchanging  $X$  and  $Y$  in this reasoning, we get maps  $a': X \rightarrow Y$  and  $b': Y \rightarrow X$  such that  $a' \circ b' \sim \text{id}_Y$ . Since  $X$  and  $Y$  are  $\ell$ -toy, these four maps are weak equivalences.  $\square$

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